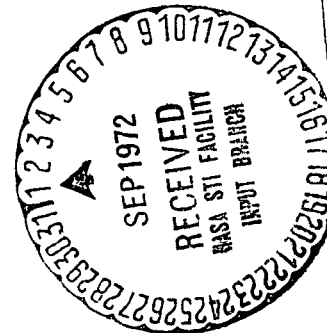


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by

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OSCILLATIONS IN NEUTRAL FUNCTIONAL DIFFERENTIAL EQUATIONS

Jack K. Hale

Some time ago [1], the author announced a result on the Fredholm alternative for the existence of periodic solutions of a non-homogeneous linear neutral functional differential equation (NFDE). In this paper, we indicate a proof of this result and, at the same time, use the method of proof to give a brief survey of some recent developments in the theory of NFDE which have applications far beyond the problem of periodic solutions.

Let $R = (-\infty, \infty)$, $R^+ = [0, \infty)$, E^n be any n -dimensional linear vector space with norm $|\cdot|$, $C([a, b], E^n)$ the space of continuous functions from $[a, b]$ to E^n with the topology of uniform convergence. For a fixed $r \geq 0$, let $C = C([-r, 0], E^n)$ with norm $|\varphi| = \sup_{-r \leq \theta \leq 0} |\varphi(\theta)|$ for $\varphi \in C$. If $x \in C([\sigma-r, \sigma+A], E^n)$ for some $A > 0$, let $x_t \in C$, $t \in [\sigma, \sigma+A)$ be defined by $x_t(\theta) = x(t+\theta)$, $-r \leq \theta \leq 0$.

Let $D, L: C \rightarrow E^n$ be continuous linear operators,

$$\begin{aligned} L\varphi &= \int_{-r}^0 [d\eta(\theta)]\varphi(\theta) \\ (1) \quad D &= D_0 + D_1 \\ D_0\varphi &= \varphi(0) - \sum_{k=1}^{\infty} A_k \varphi(-\tau_k), \\ D_1\varphi &= \int_{-r}^0 A(\theta)\varphi(\theta)d\theta \end{aligned}$$

where η is an $n \times n$ matrix function of bounded variation, $A(\theta)$ is an $n \times n$ matrix integrable on $[-r, 0]$, the A_k are $n \times n$ constant

matrices with $\sum_{\tau_k < \epsilon} |A_k| \rightarrow 0$ as $\epsilon \rightarrow 0$ and $0 < \tau_k \leq r$. For notational purposes, there is an $n \times n$ matrix μ of bounded variation on $[-r, 0]$ such that

$$(2) \quad D\varphi = \varphi(0) - \int_{-r}^0 [d\mu(\theta)]\varphi(\theta).$$

A linear homogeneous NFDE is a relation

$$(3) \quad \frac{d}{dt} Dx_t = Lx_t.$$

A solution of (3) is a continuous function x on some interval $[-r, A)$, $A > 0$, such that Dx_t is continuously differentiable and satisfies (3) on $(0, A)$. For any $\varphi \in C$, there is a unique solution $x = x(\varphi)$ of (1) on $[-r, \infty)$ such that $x_0 = \varphi$ and this solution is continuous in $(\varphi, t) \in C \times \mathbb{R}^+$ (see [2]). If $T(t, D, L): C \rightarrow C$, $t \in \mathbb{R}^+$, is defined by

$$(4) \quad T(t, D, L)\varphi = x_t(\varphi)$$

then $T(t, D, L)$, $t \in \mathbb{R}^+$, is a strongly continuous semigroup of linear transformations.

Let $C_{D_0} = \{\varphi \in C: D_0\varphi = 0\}$. This is a closed subspace of the Banach space C and $T(t, D_0, 0): C_{D_0} \rightarrow C_{D_0}$. Let the spectral radius of $T(t, D_0, 0)|_{C_{D_0}}$ be $r_{D_0}(t)$ and

$$(5) \quad a_{D_0} = \frac{1}{t} \ln r_{D_0}(t).$$

The operator D_0 is said to be stable if $a_{D_0} < 0$ (see [8]). Note that $T(t, D_0, 0) \mid C_{D_0}$ is nothing but the semigroup of linear transformations corresponding to the solutions of the homogeneous difference equation

$$(6) \quad D_0 y_t = 0.$$

The operator D_0 is stable if the zero solution of (6) is uniformly asymptotically stable. Also, note that $a_{D_0} = -\infty$ if $D_0 \varphi = \varphi(0)$; that is, D_0 is the operator corresponding to the usual retarded functional differential equations.

For a fixed $\omega > 0$, let $\mathcal{P}_\omega = \{x \in C((-\infty, \infty), E^n) : x(t+\omega) = x(t), t \in \mathbb{R}\}$ and $\mathcal{A}_\omega = \{H \in C((-\infty, \infty), E^n) : H(0) = 0 \text{ and there is an } n\text{-vector } \alpha \text{ and } h \in P_\omega \text{ with } H(t) = \alpha t + h(t)\}$. For any $H \in \mathcal{A}_\omega$, $H(t) = \alpha t + h(t)$, $h \in P_\omega$, we let $|H| = |\alpha| + \sup_{t \in [0, \omega]} |h(t)|$. The theorem on the Fredholm alternative for periodic solutions can now be stated as:

Theorem 1. If D_0 is stable, $H \in \mathcal{A}_\omega$, then the equation

$$(7) \quad \frac{d}{dt} [Dx_t - H(t)] = L(x_t)$$

has a solution in \mathcal{P}_ω if and only if

$$(8) \quad \int_0^\omega y(t) dH(t) = 0$$

for all ω -periodic row vector solutions y of the "adjoint" equation

$$(9) \quad \frac{d}{dt} [y(t) - \int_{-r}^0 y(t-\theta) d\mu(\theta)] = - \int_{-r}^0 y(t-\theta) d\eta(\theta).$$

Furthermore, there is a continuous projection operator $J: \mathcal{U}_\omega \rightarrow \mathcal{U}_\omega$ such that the set of all H satisfying (8) is $(I-J)\mathcal{U}_\omega$ and there is a continuous linear operator $\mathcal{K}: (I-J)\mathcal{U}_\omega \rightarrow \mathcal{P}_\omega$ such that $\mathcal{K}H$ is a solution of (7) for each $H \in (I-J)\mathcal{U}_\omega$.

It is part of the conclusion of Theorem 1 that the integral in (8) is well-defined even though H is only continuous.

It is also possible to explicitly describe the operator J and in doing this we will also rephrase the entire problem in terms of the solution of an operator equation in a Banach space. This terminology makes clear the relationship of the above theorem to the general problem of solving functional equations.

Let $A: \mathcal{P}_\omega \rightarrow \mathcal{U}_\omega$ be defined by

$$(10) \quad Ax(t) = Dx_t - Dx_0 - \int_0^t Lx_s ds.$$

It is clear that A is continuous, linear and, furthermore, that the null space $\mathcal{N}(A)$ consists of the solutions of the homogeneous equation (3) in \mathcal{P}_ω . It will be shown below that $\mathcal{N}(A)$ is finite dimensional and there is a continuous projection $S: \mathcal{P}_\omega \rightarrow \mathcal{P}_\omega$ such that $\mathcal{N}(A) = S\mathcal{P}_\omega$. Theorem 1 implies there is a continuous projection $J: \mathcal{U}_\omega \rightarrow \mathcal{U}_\omega$ such that the range $\mathcal{R}(A)$ satisfies $\mathcal{R}(A) = (I-J)\mathcal{U}_\omega$.

and that A has a bounded right inverse \mathcal{H} . Furthermore, \mathcal{H} will be uniquely specified if we require that $J\mathcal{H} = 0$.

If $U = (\varphi_1, \dots, \varphi_d)$ is a basis for the ω -periodic solutions of (3), $V = \text{col}(\psi_1, \dots, \psi_d)$ is a basis for the ω -periodic solutions of (9), and $'$ denotes transpose, then S, J can be defined as

$$(11) \quad \begin{aligned} Sh &= U \left[\int_0^\omega U'(s)U(s)ds \right]^{-1} \int_0^\omega U'(s)h(s)ds \\ JH(t) &= \left[\int_0^t V'(s)ds \right] \left[\int_0^\omega V(s)V'(s)ds \right]^{-1} \int_0^\omega V(s)dH(s). \end{aligned}$$

Before proceeding to the proof of the theorem, we remark that Theorem 1 allows one to immediately apply the usual theory for perturbed linear systems to equations of the form

$$(12) \quad \frac{d}{dt} [Dx_t - G(t, x_t)] = Lx_t + F(t, x_t)$$

where $G(\cdot, \varphi) \in \mathcal{X}_\omega$, $F(\cdot, \varphi) \in \mathcal{P}_\omega$ for each $\varphi \in C$. In fact, if we define the operator A as before and define $N: \mathcal{P}_\omega \rightarrow \mathcal{X}_\omega$ by

$$(13) \quad Nx(t) = G(t, x_t) - G(0, x_0) + \int_0^t F(s, x_s)ds$$

then equation (12) has a solution x in \mathcal{P}_ω if and only if

$$(14) \quad Ax = Nx$$

which by Theorem 1 and the above remarks is equivalent to

$$(15) \quad \begin{aligned} x &= Sx + \mathcal{H}(I-J)Nx \\ JNx &= 0. \end{aligned}$$

One can now apply the theory of [9] to obtain sufficient conditions for the existence of ω -periodic solutions. In particular, if $G = \epsilon \tilde{G}$, $F = \epsilon \tilde{F}$, where $\tilde{G}(t, \varphi)$, $\tilde{F}(t, \varphi)$ are continuously differentiable in φ and ϵ is a real parameter, and there is a d -vector b_0 such that

$$\begin{aligned} \Lambda(b_0) &= 0, \quad \det[\partial \Lambda(b_0) / \partial b] \neq 0 \\ \Lambda(b) &= \int_0^\omega V(s) [d\tilde{G}(s, U(s)b) + \tilde{F}(s, U(s)b) ds]. \end{aligned}$$

then there is an $\epsilon_0 > 0$ such that equation (12) has an ω -periodic solution $x(b_0, \epsilon)$, $0 \leq |\epsilon| \leq \epsilon_0$, continuous in ϵ and $x(b_0, 0) = Ub_0$.

We now proceed to give two methods of proving Theorem 1, the first method will have general applicability to the discussion of the local theory of nonlinear equations and the second is applicable to more general boundary value problems. Basic to both approaches is the following:

Lemma 1 [3]. There exists a continuous linear map $\psi: C \rightarrow C_{D_0}$ and a family of maps $T_1(t, D, L): C \rightarrow C$, completely continuous for each $t \geq 0$, such that

$$(16) \quad T(t, D, L) = T(t, D_0, 0)\psi + T_1(t, D, L).$$

From a general result on linear operators (see [4,5]), Lemma 1 implies the following: for any fixed $a > a_{D_0}$, all elements $\mu(t)$ in the spectrum $\sigma(T(t, D, L))$ of $T(t, D, L)$ with $|\mu(t)| \geq e^{at}$ belong to the point spectrum, the number of such $\mu(t)$ is finite, the generalized eigenspace of each $\mu(t)$ is finite dimensional, and there exists subspaces $P_{\mu(t)}, \tilde{P}_{\mu(t)}$ invariant under $T(t, D, L)$ with $P_{\mu(t)}$ finite dimensional, $C = P_{\mu(t)} \oplus \tilde{P}_{\mu(t)}$ and the spectrum of $T(t, D, L)$ restricted to $\tilde{P}_{\mu(t)}$ is $\sigma(T(t, D, L)) \setminus \{\mu(t)\}$.

If D_0 is stable, then, in particular, there can be only a finite number of elements of the point spectrum of $T(t, D, L)$ with moduli equal to one. Thus, there can be at most a finite number of ω -periodic solutions of (3) and this implies the existence of the projection operator P mentioned above with $\mathcal{H}(A) = P \mathcal{D}_\omega$.

Furthermore, D_0 stable implies only a finite number of elements of the point spectrum of $T(t, D, L)$ with moduli greater than or equal to one. The theory in [2] now implies one can decompose C as $C = P \oplus Q$ where P, Q are invariant under $T(t, D, L)$, $\sigma(T(t, D, L)|_Q) = \sigma(T(t, D, L)) \cap \{\lambda: |\lambda| < 1\}$, P finite dimensional and P, Q are determined in the following manner. A number λ is said to be a characteristic value of (3) if

$$(17) \quad \det \Delta(\lambda) = 0, \quad \Delta(\lambda) = \lambda \left[I - \int_{-r}^0 e^{\lambda \theta} d\mu(\theta) \right] - \int_{-r}^0 e^{\lambda \theta} d\eta(\theta).$$

Let D_0 be stable, $\Lambda = \{\lambda: \det \Delta(\lambda) = 0, \operatorname{Re} \lambda \geq 0\}$ and $\Phi = (\varphi_1, \dots, \varphi_p)$ be a basis for the initial values of all solutions of (3) of the form $\sum p_k(t)e^{\lambda_k t}$, $\lambda_k \in \Lambda$, $p_k(t)$ polynomials in t , and $\Psi = \operatorname{col}(\psi_1, \dots, \psi_p)$ a basis for the corresponding solutions of the adjoint equation (9) of the form $\sum q_k(t)e^{-\lambda_k t}$, $\lambda_k \in \Lambda$, $q_k(t)$ polynomials in t . Let

$$\begin{aligned} (\alpha, \varphi) = & \alpha(0)D\varphi + \int_{-r}^0 \int_0^\theta \alpha(\xi - \theta) [d\mu(\theta)] \varphi(\xi) d\xi \\ & - \int_{-r}^0 \int_0^\theta \alpha(\xi - \theta) [d\eta(\theta)] \varphi(\xi) d\xi \end{aligned}$$

and $(\Psi, \Phi) = (\psi_j, \varphi_k)$, $j, k = 1, 2, \dots, p$. Then (Ψ, Φ) is nonsingular and may be taken to be the identity. It follows from [2] that

$$C = P \oplus Q$$

$$(18) \quad P = \{\varphi \in C: \varphi = \Phi a \text{ for some } p\text{-vector } a\}$$

$$Q = \{\varphi \in C: (\Psi, \varphi) = 0\}.$$

Thus, any $\varphi \in C$ can be written as $\varphi = \varphi^P + \varphi^Q$, $\varphi^P = \Phi(\Psi, \varphi)$, $\varphi^Q = \varphi - \varphi^P \in Q$.

Since P is invariant under $T(t, D, L)$, there is a $p \times p$ constant matrix E , $\sigma(E) = \Lambda$ such that $T(t, D, L)\varphi = \Phi e^{Et}$. Also, from [2], $\Phi(\theta) = \Phi(0)e^{E\theta}$, $-r \leq \theta \leq 0$.

Another fact that is needed is the variation of constants formula. It is shown in [6] that there is an $n \times n$ matrix $X(t)$, $-\infty < t < \infty$, of bounded variation, continuous from the right, $X(t) = 0$, $t < 0$, $X(0) = I$ satisfying

$$D(X_t) = \int_0^t L(X_s) ds + I, \quad t \geq 0$$

such that, if $X_t \stackrel{\text{def}}{=} T(t)X_0$, the initial value problem for (7) is equivalent to

$$(19) \quad x_t = T(t)\varphi + \int_0^t T(t-s)X_0 dH(s).$$

If one now defines $X_0^P = \Phi(\Psi, X_0) = \Phi\Psi(0)$, $X_0^Q = X_0 - X_0^P$, then $x_t = \Phi y(t) + x_t^Q$ implies

$$(20) \quad \begin{aligned} \frac{d}{dt} [y(t) - \Psi(0)H(t)] &= Ey(t) \\ x_t^Q &= T(t, D, L)\varphi^Q + \int_0^t T(t-s, D, L)X_0^Q dH(s) \end{aligned}$$

where all integrals are to be interpreted as regular integrals in E^n for each θ in $[-r, 0]$.

Since the spectrum of $T(t, D, L)|_Q$ lies inside the unit circle, there are positive constants K, α such that $|T(t, D, L)|_Q| \leq Ke^{-\alpha t}$, $t \geq 0$. Therefore, it follows from [6] that one can also suppose K, α are such that $|T(t, D, L)X_0^Q| \leq Ke^{-\alpha t}$, $t \geq 0$. Thus, there is a unique $\varphi^Q = \varphi^Q(H)$ continuous and linear in H such that the solution x_t^Q of the second equation in (20) is ω -periodic in t . If $H(t) = \alpha t + h(t)$, $\alpha \in E^n$, $h \in \mathcal{D}_\omega$, and $z(t) = y(t) - \Psi(0)h(t)$ in the first equation of (20), then the existence of an ω -periodic solution is equivalent to the existence of an ω -periodic solution of the ordinary

differential equation

$$(21) \quad \frac{dz(t)}{dt} = Ez(t) + EY(0)h(t) + Y(0)\alpha.$$

A necessary and sufficient condition for the existence of an ω -periodic solution of (21) is well-known from ordinary differential equations. Checking this condition and proceeding as in [10] for retarded functional differential equations, one completes the proof of Theorem 1.

A second proof can be obtained from the general theory of two point boundary-value problems developed in [7]. We summarize the theory for the autonomous case although the nonautonomous case is also treated in [7].

For the statement of the principal results on boundary value problems, some care is needed in the specification of the continuity properties of the functions η, μ in (1), (2). Without loss in generality, one can suppose both η, μ are continuous from the left on $(-r, 0)$, vanish on $[0, \infty)$ and are equal to their values at $\theta = -r$ on $(-\infty, -r]$. Let B_0 be the space of functions $\psi: [-r, 0] \rightarrow E^{n*}$ (the space of n -dimensional row vectors) which are of bounded variation on $[-r, 0]$, continuous from the left on $(-r, 0)$ and $\psi(0) = 0$. We identify B_0 with the conjugate space of C with the pairing

$$\langle \psi, \varphi \rangle = \int_{-r}^0 [d\psi(\theta)]\varphi(\theta), \quad \psi \in B_0, \varphi \in C.$$

With μ, η normalized as above, the adjoint equation (9) as

an equation for functions of bounded variation can be written as

$$(22) \quad y(s) - \int_s^{\infty} [dy(\alpha)]\mu(s-\alpha) + \int_s^{\infty} y(\alpha)\eta(s-\alpha)d\alpha = \text{constant}.$$

It is then not difficult to prove that for any $t \in R$, $\psi \in B_0$, there is a unique $y: R \rightarrow E^{n*}$ of bounded variation on finite intervals, continuous from the left such that $y_t = \psi$, y vanishes on $[t, \infty)$ and (22) holds for $s \leq t - r$. If this solution is denoted by $y(t, \psi)$ and $y_s^0(\theta) = y(s+\theta)$, $-r \leq \theta < 0$, $y_s^0(0) = 0$, then $y_s^0(t, \psi) \in B_0$ for each $s \leq t$.

Let $\Omega: B_0 \rightarrow B_0$ be the quasinilpotent operator defined by

$$(23) \quad \Omega\psi(\theta) = -\int_{\theta}^0 [d\psi(\beta)]\mu(\theta-\beta) + \int_{\theta}^0 \psi(\beta)\eta(\theta-\beta)d\beta, \quad -r \leq \theta \leq 0, \quad \psi \in B_0.$$

Suppose V is a Banach space, $\sigma < \tau$ are given real numbers $M, N: C \rightarrow V$ are linear operators with domain dense in C and $\gamma \in V$ is fixed. Let V^* be the conjugate space of V and M^*, N^* the adjoint operators of M, N , respectively. The boundary value problem (I) is to find a solution of (7) satisfying

$$(24) \quad Mx_{\sigma} + Nx_{\tau} = \gamma.$$

Theorem 2. For boundary value problem (I) to be solvable, it is necessary that

$$(25) \quad \int_{\sigma}^{\tau} y(s) dH(s) = -\langle \psi, r \rangle_V$$

for all solutions y, ψ of the adjoint problem: $\psi \in V^*$, y satisfies the adjoint equation (22) on $[\sigma-r, \tau-r]$ and $z_{\sigma}^0 = (I+\Omega)^{-1} M^* \psi$, $z_{\tau}^0 = (I+\Omega)^{-1} N^* \psi$.

If $\mathcal{R}(M+NT(\tau-\sigma))$ is closed in V , this condition is both necessary and sufficient.

To apply this result to the proof of Theorem 1, let $V = C$, $M = -N = I$, $r = 0$, $\sigma = 0$, $\tau = \omega$. The boundary value problem (I) is then to find a solution of (7) with $x_0 = x_{\omega}$. To show $\mathcal{R}(I-T(\omega, D, L))$ is closed observe from Lemma 1, that $I - T(\omega, D_0, 0)\psi$ has an inverse so that

$$I - T(\omega, D, L) = [I - T(\omega, D_0, 0)\psi] [I - (I - T(\omega, D_0, 0)\psi)^{-1} T_1(\omega, D, L)].$$

Since $T_1(\omega, D, L)$ is completely continuous, it follows that this range is closed. Thus, (8) is necessary and sufficient for the existence of an ω -periodic solution of (7). This characterizes the range of the operator A in (10). Since it also is shown in [7] that the dimension of the space of ω -periodic solutions of the adjoint equation is the same as the dimension of the space of ω -periodic solutions of (3) and, thus, is finite, there exists a continuous projection operator $J: \mathcal{X}_{\omega} \rightarrow \mathcal{X}_{\omega}$ such that $\mathcal{R}(A) = (I-J)\mathcal{X}_{\omega}$. It follows that A has a bounded right inverse \mathcal{K} and Theorem 1 is proved.

Incidentally, the above argument applies equally as well to show that $M + NT(\omega, D_0, 0)$ nonsingular implies $\mathcal{R}(M + NT(\omega, D, L))$ is closed.

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